THE VON NEUMANN ALGEBRA OF THE CANONICAL EQUIVALENCE RELATION OF THE GENERALIZED THOMPSON GROUP

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Abstract. We study the equivalence relation R_N generated by the (non-free) action of the generalized Thompson group F_N on the unit interval. We show that this relation is a standard, quasipreserving ergodic equivalence relation. Using results of Feldman-Moore, Krieger and Connes we prove that the von Neumann algebra $M(R_N)$ associated to R_N is the hyperfinite type III_λ factor, with $\lambda = 1/N$.
Moreover we analyze R_N and $F(N)$ in connection with Gaboriau's work on costs of groups. We prove that the cost $C(F(N)) = 1$ for any $N \geq 2$ and for $N = 2$ we precisely find a treeing of R_N.

1. Introduction

In the following we prepare the definitions we need in this paper. We also mention some known results we are going to use: we follow [Gab], [FMII], [Br] and [Can].
We say that R is a SP1 equivalence relation on a standard probability space (X, λ) if

(S) Almost each orbit $R[x]$ is at most countable and R is a Borel subset of $X \times X$.

(P) For any $T \in \mathrm{Aut}(X, \lambda)$ such that graph$T \subset R$ we have that T preserves the measure λ.

We say that R is standard if only (S) is satisfied. Also, R is called quasi-preserving if the saturation (through R) of a null set is null.
From now on, unless specified otherwise, each equivalence relation satisfies (S). Next we define "graphing" and "treeability" with respect to R. This is just a simple adaptation of the SP1 situation (see [Gab]).

Definition 1.1. i) A countable family $\Phi = (\varphi_i : A_i \to B_i)_{i\in I}$ of Borel partial isomorphisms between Borel subsets of (X, λ) is called a graphing on (X, λ) (we do not require that the φ_i's preserve λ).
ii) The equivalence relation R_Φ generated by a graphing Φ is the smallest equivalence relation S such that $(x, y) \in S$ iff x is in some A_i and $\varphi_i(x) = y$.
iii) An equivalence relation R is called treeable if there is a graphing Φ such that $R = R_\Phi$ and almost every orbit $R_\Phi[x]$ has a tree structure. In such case Φ is called a treeing of R.
iv) R is ergodic iff any saturated Borel set has measure 0 or 1.

Remark 1.2. For (SP1) R's the same notions are considered in [Gab] provided the φ_i's preserve the measure. One can consider the quantity $C(\Phi) = \sum \lambda(A_i)$. The cost of a (SP1) equivalence relation will simply be

$$C(R) = \inf\{C(\Phi) | \Phi \text{ is a graphing of } R\}.$$

It is the preserving property that allows one to conclude the infimum is attained iff R admits a treeing (see Prop.I.11 and Thm.IV.1 in [Gab]). Next Gaboriau defines the cost of a discrete countable group G as

$$C(G) = \inf\{C(R)|R \text{ coming from a free, preserving action of } G \text{ on } X\}.$$

We highlight the following result that "measures" the non-amenability of cost 1 groups (any amenable group has cost=1):

Theorem([Gab], Corollaire VI.22) *Any non − amenable cost* 1 *group is anti − treeable* (i.e., any SP1 equivalence relation coming from a free action is not treeable).

Among many examples of groups whose costs are calculated, the Thompson group is shown to have cost=1 (using the infinite presentation of the group and one of the tools developed by Gaboriau). Now any countable discrete group comes with a free preserving action on some standard probability space, namely the Bernoulli shifts (thus the infimum in $C(G)$ does make sense). However, to handle (in terms of (non)treeability) the SP1 relation determined by this purely theoretical action may be very hard. Until we find a suitable action of the generalized Thompson group, we are content to study its canonical action on $([0, 1], \lambda)$, where λ denotes the Lebesgue measure. Certainly this is not a (SP1) relation but it is (S) and quasi-preserving.

Let us introduce some basics facts about the Thompson groups.

Definition 1.3. The Thompson group F is the set of piecewise linear homeomorphisms from the closed unit interval $[0, 1]$ to itself that are differentiable except at finitely many dyadic rationals and such that on intervals of differentiability the derivatives are powers of 2.
If $N \geq 2$, replacing above the dyadic rationals by N-adic rationals and the power of 2 slopes by powers of N, we obtain one of the generalized versions of the Thompson group. We will denote it by $F(N)$.

Remark 1.4. It is shown that $F(N)$ above is a countable subgroup of the group of all homeomorphisms from $[0, 1]$ to $[0, 1]$. Two presentations of F are found. One finite presentation comes from the fact that F is generated by the functions A and B defined below

$$A(x) = \begin{cases} x/2, & 0 \leq x \leq 1/2 \\ x - 1/4, & 1/2 \leq x \leq 3/4 \\ 2x - 1, & 3/4 \leq x \leq 1 \end{cases}, \quad B(x) = \begin{cases} x, & 0 \leq x \leq 1/2 \\ x/2 + 1/4, & 1/2 \leq x \leq 3/4 \\ x - 1/8, & 3/4 \leq x \leq 7/8 \\ 2x - 1, & 7/8 \leq x \leq 1 \end{cases}$$

The relations between generators A and B are $[AB^{-1}, A^{-1}BA] = 1$ and $[AB^{-1}, A^{-2}BA^2] = 1]$. $F(N)$ has also a finite presentation, see [Br]. However, for computing the cost of $F(N)$ we will make use of the following infinite presentation

$$F(N) = \langle x_0, x_1, ...x_i, ...|\ x_jx_i = x_ix_{j+N-1},\ i < j\ \rangle$$

Next we will introduce the von Neumann algebra of an equivalence relation. We follow [FMII] in the particular case when the 2-cocycle σ is trivial. Let R be a standard equivalence relation on the standard probability space (X, λ). The Hilbert space the algebra acts upon is $H := L^2(R, \nu_r)$ where ν_r is the right-counting

measure on R. When no confusion with the left-counting measure ν_l may arise, we will write ν instead of ν_r. E.g., if $f \in H$, its squared norm is given by

$$\int |f(x,y|^2 d\nu(x,y) = \int (\sum_{(z,x)\in R} |f(x,z)|^2) d\lambda(x)$$

For a left-finite function $a : R \to \mathbb{C}$, we denote by L_a the bounded operator

$$L_a\varphi(x,y) = \sum_z a(x,z)\varphi(z,y).$$

Then $M(R)$ is defined as $\{L_a | a \text{ is left-finite }\}''$. It is known that $L^\infty(X)$ can be embedded as a Cartan subalgebra into $M(R)$. Also, φ_0 the characteristic function of the diagonal in R is a separating and cyclic vector. Any element $L \in M(R)$ can be written as L_ψ (where $\psi = L\varphi_0$), meaning that

$$L\varphi(x,y) = \sum_z \psi(x,z)\varphi(z,y)$$

for all $\varphi \in H$ and all $(x,y) \in R$. Now the multiplication on $M(R)$ can be written as a convolution over R: $L_{\psi_1} * L_{\psi_2} = L_{\psi_1 * \psi_2}$ where

$$\psi_1 * \psi_2(x,y) = \sum_z \psi_1(x,z)\psi_2(z,y), \quad (x,y) \in R.$$

Moreover, if R is ergodic then $M(R)$ is a factor.
If the equivalence relation R is coming from the action of a discrete countable group G on the probability space (X,λ), then $M(R)$ is the crossed product of $L^\infty(X)$ by G. This is exactly the situation we will work in, however we prefer the Feldman-Moore setting.

It is easy to show that if the measure is R-invariant (i.e., R satisfies (P) above) then the state $< \cdot\varphi_0, \varphi_0 >$ is a trace; in this case $M(R)$ is a factor of type II. If there is no σ-finite measure μ, R-invariant such that $\mu \prec \lambda$ then by Theorem 2.4 in [Kr], $M(R)$ has to be of type III. (The non-existence of such μ proves that R is of type III, see the terminology in [FMI]). This is the result we are going to use for the canonical R_N on $F(N)$. Also, it turns out that the Connes spectrum has a nice description for factors coming from ergodic equivalence relations, namely the asymptotic range of the map $D : R \to \mathbb{R}_+$

$$D(x,y) = \frac{\partial \nu_l}{\partial \nu_r}(x,y)$$

therefore, to get the type of the factor suffices to "compute" the values of D. Following [Co] we obtain that $M(R_N)$ is the crossed-product of the hyperfinite II_∞ factor by $\mathbb{Z}$.

2. $M(R_N)$

In the following if a "measurable" statement is made with respect to points on the real line (or plane) then it is understood that the measure taken into account is the Lebesgue measure.

Definition 2.1. The equivalence relation $R_N \subset [0,1]\times[0,1]$ defined by $(x,y) \in R_N$ iff there exists $f \in F(N)$ such that $f(x) = y$ is called the canonical equivalence relation of the generalized Thompson group.

Remark 2.2. We can work with a subrelation of R_N (still denoted R_N) which is R_N except a set of (product) measure 0. This change will not affect the (S) or (P) properties nor the construction of $M(R_N)$. In our case R_N is replaced by R_N minus the points of rational coordinates.

Remark 2.3. We pause for a moment to distinguish the situations $N = 2$ and $N > 2$. Let R be the equivalence relation generated by the $ax + b$ group with a of the form 2^n, $n \in \mathbb{Z}$ and $b \in \mathbb{R}$, dyadic, i.e. $b = k/2^m$ for some $k, m \in \mathbb{Z}$. Notice that R_2 is the restriction of R to the unit square (however, this is not obvious: given $y = ax + b$ one has to construct $f \in F$ such that $f(x) = y$ and this can be carried-out by using the properties of the Thompson group, see Lemma 4.2 in [Can]). Interestingly enough, a similar "localization" for R_N with odd N fails to hold true: we prove that for any $x \in [0, 1] \setminus \mathbb{Q}$ there exists no $f \in F(N)$ such that $f(x) = x + \frac{k}{N^p}$ where k is odd and $p \geq 0$.

First, take x N-adic, $x = \frac{a}{N^r}$ with $r \geq p$. Then

$$x + \frac{k}{N^p} = \frac{a + kN^{r-p}}{N^r} =: \frac{b}{N^r},$$

and observe that a and b have different parity.

Assume now that there is $f \in F(N)$ such that $f(x) = x + \frac{k}{N^p}$. Consider all the points of non-differentiability $\{l_i/N^{s_i} \,|\, i \in \mathcal{I}\}$ of f and take $r \geq \max\{s_i \,|\, i \in \mathcal{I}\}$.

Let N^{q_i} ($q_i \in \mathbb{Z}$) be the slope of f on the interval $[i/N^r, (i+1)/N^r]$, ($i \in \{0, ..., N^r - 1\}$). Then

$$f\left(\frac{i+1}{N^r}\right) = f\left(\frac{i}{N^r}\right) + N^{q_i}\frac{1}{N^r}, \quad (i \in \{0, ..., N^r - 1\}).$$

By induction, since $f(0) = 0$ we obtain that

$$f\left(\frac{k}{N^r}\right) = \sum_{i=0}^{k-1} N^{q_i}\frac{1}{N^r}, \quad (k \in \{1, ..., N^r\}).$$

In particular

$$\frac{b}{N^r} = f\left(\frac{a}{N^r}\right) = \sum_{i=0}^{a-1} N^{q_i}\frac{1}{N^r}$$

Now take $q \geq \max\{-q_i \,|\, i \in \{1, ..., a-1\}\}$ and multiply by N^q:

$$bN^q = \sum_{i=0}^{a-1} N^{q+q_i}. \tag{2.1}$$

Since a and b have different parity and since N is odd, it follows that the terms of the equality (2.1) have different parity. This is a contradiction which shows that d is not equivalent to $d + k/N^p$ for d N-adic.

Now, if x is not rational, assume that x is equivalent to $x + k/N^p$. This means that there is an $f \in F(N)$ such that $f(x) = x + k/N^p$. Take a small interval around x where f is differentiable. On this interval f has the form $f(y) = N^s y + e$ with $s \in \mathbb{Z}$ and e N-adic. But then $s = 0$ and $e = k/N^r$ otherwise $x + k/N^r = N^s x + e$ and this would imply that x is rational. So on this interval $f(y) = y + k/N^p$. We can find an N-adic point in this interval, call it d, such that $f(d) = d + k/N^p$ and this contradicts the fact that d and $d + k/N^p$ are not equivalent.

We have to make sure that R_N is standard. The finite presentation of $F(N)$ implies that it is quasi-preserving. It is not hard to see that the group $F(N)$ is at most countable: given x_1, x_2..., x_k a list of N-adic points in $[0,1]$ and a list of power of N slopes there can be at most one element $f \in F(N)$ that fullfils these data. Therefore $F(N)$ is at most countable. We will actually show it is countable by displaying a non-trivial element in $F(N)$, useful also in the proofs below.

Proposition 2.4. *Let d a N-adic in $[0,1]$ and $p \in \mathbb{Z}$ such that $d < N^p$. Then the following function is an infinite order element of $F(N)$:*

$$A_{d,p}(x) = \begin{cases} x/N^p, & 0 \le x \le d \\ x - d + d/N^p, & d \le x \le 1 - d/N^p \\ N^p x + 1 - N^p, & 1 - d/N^p \le x \le 1 \end{cases}$$

Proof. The way $A_{d,p}$ is defined shows that it is an element of $F(N)$. Also, $A_{d,p} \neq$id, therefore all its iterates are distinct elements of $F(N)$. □

We will show that the von Neumann algebra $M(R_N)$ is the type $III_{1/N}$ hyperfinite factor. We first prove ergodicity in order to insure that we are dealing with a factor.

Proposition 2.5. *The equivalence relation S_N defined on [0,1] by $(x,y) \in S_N$ iff there exists $f \in F(N)$ such that $f(x) = y$ and $f'(x) = 1$, is an ergodic subrelation of R_N. Moreover, S_N is a (SP1) hyperfinite equivalence relation with infinite orbits.*

Proof. Notice that if $(x,y) \in S_N$ through some $f \in F(N)$ then $f' = 1$ on a neighborhood of x (x not being N-adic). Clearly $S_N \subset R_N$. Let now X be a S_N-saturated set. We show that for any $0 < d_1 < d_2 < 1$ N-adic numbers the following equality holds:

$$\lambda(X \cap [d_1, d_2]) = \lambda(X \cap [0, d_2 - d_1]) \tag{2.2}$$

Choose $p \in \mathbb{N}$ large enough such that $d_2 < 1 - d_1/N^p$. Because $[d_1, d_2] \subset [d_1, 1 - d_1/N^p]$ and $A_{d_1,p}$ has slope 1 on $[d_1, 1 - d_1/N^p]$ we have

$$\lambda(X \cap [d_1, d_2]) = \lambda(A_{d_1,p}(X \cap [d_1, d_2])) = \lambda(A_{d_1,p}(X) \cap [d_1/N^p, d_2 - d_1 + d_1/N^p])$$

We prove

$$A_{d_1,p}(X) \cap [d_1/N^p, d_2 - d_1 + d_1/N^p] = X \cap [d_1/N^p, d_2 - d_1 + d_1/N^p]$$

Let $y \in A_{d_1,p}(X) \cap [d_1/N^p, d_2 - d_1 + d_1/N^p]$. Then $y = A_{d_1,p}(x)$ for some $x \in X \cap [d_1, d_2]$. But on $[d_1, d_2]$ the slope of $A_{d_1,p}$ is 1, therefore $(x,y) \in S$. We get $y \in X$ from the fact that $x \in X$ and X is saturated. Vice-versa, let $y \in X \cap [d_1/N^p, d_2 - d_1 + d_1/N^p]$. Then $y = A_{d_1,p}(x)$ where $x = A^{-1}_{d_1,p}(y)$ which together with X being saturated insures $x \in X$ (also, the slope of $A^{-1}_{d_1,p}$ is 1, around y). In conclusion the above sets are equal. From the last two relations we obtain

$$\lambda(X \cap [d_1, d_2]) = \lambda(X \cap [d_1/N^p, d_2 - d_1 + d_1/N^p])$$

Taking the limit when p goes to infinity we obtain (2.2). If $0 < d_1 < d_2 < d_3 < 1$ are three consecutive N-adic numbers then from (2.2) $\lambda(X \cap [d_1, d_2]) = \lambda(X \cap [d_2, d_3])$. For any $p \in \mathbb{N}$, covering the unit interval with N^p consecutive N-adic rationals we obtain

$$\lambda(X) = N^p \lambda(X \cap [d_i, d_{i+1}]) = \frac{\lambda(X \cap [d_i, d_{i+1}])}{\lambda([d_i, d_{i+1}])}$$

where $d_{i+1} - d_i = 1/N^p$. Suppose now $\lambda(X) > 0$. Then there exists $x \in X$ a Lebesgue point. For any p we can find a sequence $(\frac{k_p}{N^p})_{p>0}$ of N-adic such that $x \in \cap_{p>0}[\frac{k_p}{N^p}, \frac{k_p+1}{N^p}]$. Hence $N^p\lambda(X \cap [k_p/N^p, (k_p+1)/N^p]) \to 1$ when $p \to \infty$. This together with the last equality implies $\lambda(X) = 1$. In conclusion S_N is ergodic.

Let S be the equivalence relation determined by the N-adic translations modulo the unit interval, i.e. $(x, y) \in S$ iff $|x - y| = d$ for some N-adic $d \in [0, 1]$ (by remark 2.3, S is not included in R_N). Notice that if $(x, y) \in S_N$ then $f(x) = y$ with $f'(x) = 1$ and $f \in F(N)$. This implies $f(x) = x + d$ for d N-adic, therefore $(x, y) \in S$. Because S is hyperfinite we obtain S_N hyperfinite: indeed, write the equivalence class $S[x] = \cup_n R_n[x]$ where $(R_n)_n$ is an increasing sequence of finite equivalence relations. Then $(S_N \cap R_n)_n$ is an increasing sequence of finite equivalence relations. We argue that S_N is with infinite orbits: let $x \in [0, 1]$ and $d < x$ a N-adic. For all sufficiently large p we have $A_{d,p}(x) \in S_N[x]$. Now, if $A_{d,p_1}(x) = A_{d,p_2}(x)$ then, as p_1 and p_2 are large enough we have $x - d + d/N^{p_1} = x - d + d/N^{p_2}$ hence $p_1 = p_2$. In conclusion the S_N-orbit of x is infinite. □

Let ϕ be the faithful normal state determined by the scalar product with φ_0. Recall the definition of the centralizer
$M^\phi = \{L \in M(R) : \phi(LT) = \phi(TL),\ \forall\, T \in M(R)\}$. We know from [Co] that for III_λ factors there exists a faithful normal state such that the centralizer is a factor of type II_1. We are now ready to prove the main result of this section.

Theorem 2.6. *The von Neumann algebra $M(R_N)$ is the hyperfinite factor of type $III_{1/N}$. The core M^ϕ is the hyperfinite II_1 factor isomorphic to $M(S_N)$.*

Proof. The above proposition shows that R_N is ergodic as well, therefore $M(R_N)$ is a factor. Suppose now that there exists a σ-finite measure μ, R_N-invariant such that $\mu \prec \lambda$. We take the Radon-Nikodym derivative $f := \frac{\partial \mu}{\partial \lambda}$. By invariance of μ with respect to R_N and with a substitution we obtain

$$f(x) = f(T(x))T'(x),\ \forall\, T \in F(N), \text{a.e.} x \in [0, 1] \tag{2.3}$$

For some fixed values $a < b$, we consider the set
$A := \{x \in [0, 1] \mid f(x) \in [a, b]\}$. We show that A is S_N-saturated: if $(x, y) \in S_N$ with $x \in A$ then there is a $T \in F(N)$ such that $T(x) = y$ and $T'(x) = 1$. Applying equation (2.3) we get $f(y) = f(x)$, therefore $y \in A$. By ergodicity A has to be of Lebesgue measure 0 or 1. Because a and b are arbitrary we obtain that f must be constant. This is not possible though, as the Lebesgue measure λ is not R_N-invariant. In conclusion there is no such measure μ, therefore $M(R_N)$ is a factor of type III.

We prove next that $M(R_N)$ is of type $III_{1/N}$. We use Proposition 2.2 in [FMI]: in particular it says that for $T \in F$ we have $D(T^{-1}(y), y) = dT_*(\lambda)/d\lambda(y)$ a.e. y. Thus for any Borel subset A of $[0, 1]$ we have

$$\int_A (T^{-1})'(y)dy = \lambda(T^{-1}(A)) = \int_A D(T^{-1}(y), y)dy$$

Therefore

$$\forall\, T \in F : D(x, T(x)) = \frac{1}{T'(x)},\ \text{a.e.} x \tag{2.4}$$

The above equation (almost) finds the range, $N^{\mathbb{Z}}$, of the map $D : R \to \mathbb{R}_+$. Indeed, for any $(x, y) \in R$ there exists an unique $T \in F(N)$ such that $T(x) = y$ (if there

are $T_1 \neq T_2$ in $F(N)$ such that $T_1(x) = T_2(x)$, then x must be N-adic rational, a value which we avoid by remark 2.2). We will actually compute the asymptotic range of D,

$$r^*(D) = \{a | \; \forall \; V_a \text{ neighborhood of } a \; \forall \; Y \subset [0,1] \text{ of positive measure } : \\ \text{pr}\{(x,y) \in Y \times Y \mid D(x,y) \in V_a\} = Y \text{ a.e. } \}$$

Notice first that
$\text{pr}\{(x,y) \in Y \times Y \mid D(x,y) \in V_a\} = \{x \in Y \mid \exists \; T \in F(N){:}D(x,T(x)) \in V_a\}$
If $a \notin N^{\mathbb{Z}}$ then there is a neighborhood V_a of a such that $V_a \cap N^{\mathbb{Z}} = \emptyset$. This combined with $D(x,T(x)) \in N^{\mathbb{Z}}$ and equation (2.4) implies $\lambda\{x \in Y \mid \exists \; T \in F{:}D(x,T(x)) \in V_a\} = 0$, which means $a \notin r^*(D)$.
For proving the other inclusion, let $p \in \mathbb{Z}$ and $Y \subset [0,1]$ with $\lambda(Y) > 0$. From the definition of the asymptotic range of the map D, suffices to show: a.e. $x \in Y$, $\exists$ $y \in Y$, $\exists \; T \in F(N)$ such that $T(x) = y$ and $T^{'}(x) = N^{-p}$ (because $D(x,T(x)) = 1/T^{'}(x)$). For $S_N[Y]$, the saturation of Y through S_N, we have $\lambda(S_N[Y]) = 1$. Consider the set

$$Y_1 := \{y \mid \exists \; x \in Y, \; \exists T \in F(N) \text{ such that } T(x) = y, T^{'}(x) = N^{-p}\}$$

Then $\lambda(Y_1 \setminus S_N[Y]) = 0$. Because $F(N)$ is countable and its elements preserve the null sets the following set is of measure 0,

$$C := \bigcup_{T \in F(N)} T^{-1}(Y_1 \setminus S[Y])$$

Now, let $x \notin C$ and $x \in Y$. Choose $0 < d < 1$ a N-adic such that $x \in [0,d]$. Then for $T_1 := A_{d,p}$ we have $T_1^{'}(x) = N^{-p}$. The point x not being in C we obtain $T_1(x) \in S_N[Y]$, i.e. $\exists \; T_2$ with $T_2^{'}(T_1(x)) = 1$ and $T_2(T_1(x)) \in Y$; the point $y := T_2(T_1(x))$ is the one we are looking for. Therefore $M(R_N)$ is a type $III_{1/N}$ factor. To check the last part of the theorem notice that the kernel of the Radon-Nikodym derivative D equals precisely S_N. From here it is rather standard ([Co], [Ta]) to conclude that the core M^{ϕ} is $M(S_N)$ and $M(R_N)$ is the crossed-product of the hyperfinite II_{∞} factor by $\mathbb{Z}$. □

Remark 2.7. For the particular case when $N = 2$, Sergey Neshveyev pointed out that we can show that $M(R_2)$ is the hyperfinite $III_{1/2}$ factor in the following way:

Consider the $ax + b$ group with a of the form 2^n, $n \in \mathbb{Z}$ and $b \in \mathbb{R}$, dyadic, i.e. $b = k/2^m$ for some $k, m \in \mathbb{Z}$. The multiplication is given by $(a,b)(a',b') = (aa', ab' + b)$. This groups acts naturally on $\mathbb{R}$ by dyadic translations and dilations by powers of 2. It therefore generates an equivalence relation on $\mathbb{R}$. From remark 2.3 the restriction of this equivalence relation to $[0,1]$ is R_2.

The crossed-product $M_{\mathbb{R}}$ of $L^{\infty}(\mathbb{R})$ with this action decomposes as follows: first, the dyadic translations act freely and ergodically on $L^{\infty}(\mathbb{R})$, so that the crossed-product is a hyperfinite II_{∞} factor. Then the dilations by 2 induce an automorphism on this II_{∞} factor that scales the semi-finite trace by 2. Therefore, using Connes results [Co], we get that $M_{\mathbb{R}}$ is a hyperfinite $III_{1/2}$ factor. Now take the projection p given by the characteristic function of $[0,1]$. The compression $pM_{\mathbb{R}}p$ is isomorphic to $M_{\mathbb{R}}$ (since we are in a type III factor); on the other hand, it can be shown that this compression is isomorphic to our $M(R_2)$.
Notice that the same "compression" argument cannot work for general N, see the counterexample in remark 2.3.

3. A treeing of R_2

Let us notice that R_N is treeable being a hyperfinite equivalence relation. This is a consequence of the general theory developed mainly by H.Dye, W.Krieger and Connes-Feldman-Weiss. In the following using the finite generation of F we will precisely find such a treeing.
Let A and B the piecewise linear homeomorphisms that generate F. Let us consider the following graphing:
$\Phi = (\varphi_i : A_i \to B_i)_{i\in\{1,2,3\}}$ where φ_i's are defined as follows:
$\varphi_1 : [0, 1/2] \to [0, 3/4]$, $\varphi_1(x) = A^{-1}(x)$,
$\varphi_2 : [1/2, 3/4] \to [1/2, 7/8]$, $\varphi_2(x) = B^{-1}(x)$,
$\varphi_3 : [3/4, 1] \to [1/2, 1]$, $\varphi_3(x) = A(x)$.

Proposition 3.1. $R = R_\Phi$

Proof. Clearly $R_\Phi \subset R$. Let $(x, y) \in R$ i.e. $\omega(x) = y$ for $\omega \in F$ word over the letters A, A^{-1}, B, B^{-1}. Notice that suffices to show $(x, y) \in R_\Phi$ for $\omega \in \{A, B\}$ (apply induction on the length of ω).
Case I: $A(x) = y$
I.1: If $x \in [0, 1/2]$ then $A(x) = x/2 = y \in [0, 1/4] \subset [0, 1/2]$, hence $x = \varphi_1(y)$
I.2: If $x \in [1/2, 3/4]$ then $A(x) = y = x - 1/4 \in [1/4, 1/2] \subset [0, 1/2]$, hence $x = \varphi_1(y)$
I.3: If $x \in [3/4, 1]$ then $\varphi_3(x) = y$
For this case we conclude $(x, y) \in R_\Phi$.
Case II: $B(x) = y$
II.1: If $x \in [0, 1/2]$ then $x = y$
II.2: If $x \in [1/2, 3/4]$ then $y = x/2 + 1/4 \in [1/2, 5/8]$, hence $x = \varphi_2(y)$
II.3: If $x \in [3/4, 7/8]$ then $y = x - 1/8 \in [5/8, 3/4]$, hence $x = \varphi_2(y)$
II.4: If $x \in [7/8, 1]$ then $y = 2x - 1 \in [3/4, 1]$, hence $x = \varphi_3(y)$
From all cases we conclude $(x, y) \in R_\Phi$ □

Theorem 3.2. *For all ω reduced words over Φ, the set $\{x \in [0, 1] \mid \omega(x) = x\}$ has Lebesque measure zero, i.e. almost every orbit has a tree structure.*

Proof. If $\omega = \varphi_{i_1}^{\epsilon_1}\varphi_{i_2}^{\epsilon_2}...\varphi_{i_k}^{\epsilon_k}$ is a reduced word over Φ then $i_j \in \{1, 2, 3\}$, $\epsilon_j \in \{-1, 1\}$ and if $i_j = i_{j+1}$ then $\epsilon_j = \epsilon_{j+1}$. To show the set of fixed points has measure zero we use induction on the length k. The case $k = 1$ being trivial we assume for any reduced word of length $k - 1$ the measure of its fixed points is 0. Take ω of length k and x such that $\omega(x) = x$. We may discard the orbits of $x = 1/2$ and $x = 3/4$ as these are countable sets. We distinguish three cases:

I. $x \in [0, 1/2)$
We must have $i_k = 1 = i_1$, φ_1 being the only generator whose domain is $[0, 1/2]$ and that can target points in $[0, 1/2)$. If $\epsilon_1 \neq \epsilon_k$ apply the induction hypothesis for the word $\varphi_{i_2}^{\epsilon_2}...\varphi_{i_k}^{\epsilon_k}$. If $\epsilon_1 = \epsilon_k = 1$ then $\varphi_{i_2}^{\epsilon_2}...\varphi_{i_{k-1}}^{\epsilon_{k-1}}\varphi_1(x) = \varphi_1^{-1}(x) \in [0, 1/2)$. As above we obtain $i_2 = 1$. ω being reduced we have $\epsilon_1 = \epsilon_2$ so that $\varphi_{i_3}^{\epsilon_3}...\varphi_{i_{k-1}}^{\epsilon_{k-1}}\varphi_1(x) = \varphi_1^{-2}(x) \in [0, 1/2)$. Inductively we obtain all subscripts $i_j = 1$. The equation $\omega(x) = x$ becomes $\varphi_1^k(x) = x$, therefore there is at most one solution for $\omega(x) = x$. By symmetry, the case $\epsilon_1 = \epsilon_k = -1$ has a similar argument.

II. $x \in (1/2, 3/4)$
Suppose $i_k = 1$. In order $\varphi_1^{\epsilon_k}(x)$ to make sense we must have $\epsilon_k = -1$. Because ω is reduced and $\varphi_1^{-1}(x) \in [0, 1/2)$ the only choice for the letter $\varphi_{i_{k-1}}^{\epsilon_{k-1}}$ is φ_1^{-1}. Continuing this procedure we would make all letters of ω equal to φ_1^{-1}, i.e. x is a fixed point of φ_1^k. Same conclusion holds if $i_1 = 1$. Suppose now $i_1,\ i_k \in \{2, 3\}$. We distinguish the following subcases:
II.1. $\omega(x) = \varphi_2^{\epsilon_1}\overline{\omega}\varphi_2^{\epsilon_k}(x) = x$. If $\epsilon_1 \neq \epsilon_k$ then the induction hypothesis will end the proof. By symmetry suffices to check only the case $\varphi_2^{-1}\overline{\omega}\varphi_2^{-1}(x) = x$. We claim that $\overline{\omega}$ has the following Φ-writting: $\overline{\omega} = \varphi_3^{-p_1}\varphi_2^{-q_1}...\varphi_3^{-p_l}\varphi_2^{-q_l}$ where $p_j \geq 0$, $q_j \geq 0$ are integers. Because of the way we choose the domains the following statements are true (the "reading" of ω is done from right to left, i.e. letter x is after letter y in xy):
• There can be no φ_1^{ϵ} occurence in $\overline{\omega}$: inded, a φ_1 occurence will force all letters to the right of φ_1 be equal to φ_1. This in not allowed as the right-end letter takes on $x \in [1/2, 3/4]$. A φ_1^{-1} occurence is not allowed otherwise all letters to the left of it would be equal to φ_1^{-1}, including the left-end. In this case $\omega(x) = x$ would be sent in $[0, 1/2]$.
• A φ_3 occurence immediately after φ_2^{-1} is not possible.
• After a φ_2^{-1} occurence only a φ_3^{-1} or φ_2^{-1} occurence is allowed.
• A φ_2 occurence immediately after φ_3^{-1} is not possible.
• After a φ_3^{-1} occurence only a φ_3^{-1} or φ_2^{-1} occurence is allowed.
All of the above prove the claim. We show that the equation $\omega(x) = x$ has at most one solution: φ_2^{-1} takes $[1/2, 3/4]$ into $[1/2, 3/4]$ and $\varphi_2^{-1}(x) = x/2 + 1/4$ so that with each iteration the slope will decrease by a factor of 2; we apply a $1/2$ slope at least once at the right-end of ω when computing $\varphi_2^{-1}(x)$ (it may be that at some step in the composition the trajectory exits $[1/2, 3/4]$ and φ_2^{-1} takes on slope $= 1$ but the slope has already been "damaged" at the begining); the slope is decreased further by $\varphi_3^{-1} = (x + 1)/2$. Now the equation $\varphi_2^{-1}\overline{\omega}\varphi_2^{-1}(x) = x$ can be written $ax + b = x$ for some $a < 1$.
II.2. $\omega(x) = \varphi_3^{\epsilon_1}\overline{\omega}\varphi_3^{\epsilon_k}(x) = x$. Again by the induction hypothesis suffices to argue only for the case $\varphi_3^{-1}\overline{\omega}\varphi_3^{-1}(x) = x$: this is easy as φ_3^{-1} targets $[3/4, 1]$ but $x \in (1/2, 3/4)$.
II.3. $\omega(x) = \varphi_2^{\epsilon_1}\overline{\omega}\varphi_3^{\epsilon_k}(x) = x$. Because $x < 3/4:\ \epsilon_k = -1$. A similar analysis of occurences and slopes< 1 leads to an equation with one solution at most.
II.4. $\omega(x) = \varphi_3^{\epsilon_1}\overline{\omega}\varphi_2^{\epsilon_k}(x) = x$. This case is symmetric to II.3.

III. $x \in (3/4, 1]$
Again we discard φ_1^{ϵ}'s occurences in ω: a φ_1 occurence will force all letters to the right of φ_1 be equal to φ_1. This in not allowed as the right-end letter takes on $x \in (3/4, 1]$. A φ_1^{-1} occurence is not allowed otherwise all letters to the left of it would be equal to φ_1^{-1}, including the left-end. In this case $\omega(x)$ would be sent in $[0, 1/2]$. We list now all possibilities for the first and last letter of ω:
$\varphi_{i_1}^{\epsilon_1} \in \{\varphi_2,\ \varphi_3^{\epsilon}\}$, $\varphi_{i_k}^{\epsilon_k} \in \{\varphi_2^{-1},\ \varphi_3^{\epsilon}\}$ where $\epsilon \in \{-1, 1\}$.
The cases $\omega \in \{\varphi_3^{-1}\overline{\omega}\varphi_3,\ \varphi_2\overline{\omega}\varphi_2^{-1},\ \varphi_3\overline{\omega}\varphi_3^{-1}\}$ can be dealt with by the induction hypothesis. All the other remaining cases can be dealt with by the same analysis of occurences in II.1 : e.g. if $\omega = \varphi_3^{-1}\overline{\omega}\varphi_3^{-1}$ then the first letter (from the right) of $\overline{\omega}$ is either φ_3^{-1} or φ_2^{-1} etc; in the end $\overline{\omega}$ becomes a word written with iterates of φ_3^{-1}

and/or φ_2^{-1}. Because φ_3^{-1} has slope 1/2 and φ_2^{-1} has slope 1 or 1/2 we conclude that the equation $\omega(x) = x$ is equivalent to $ax + b = x$ with $a < 1$.
With the analysis of I, II and III we complete the k^{th} step of induction, thus proving the theorem. $\square$

Remark 3.3. Using the infinite presentation of the Thompson group F it can be shown $C(F) = 1$. Using Gaboriau's results we will describe how to compute this cost, but for the general version $F(N)$. Still, the question is whether the cost of the normal subgroup $[F, F]$ is 1 or > 1(in this case F would be non-amenable); we believe it should be 1, even though we do not know if the following procedure can be carried-out for $[F, F]$ instead.

The following properties are easy to work-out:
i) any non-trivial element of $F(N)$ is of infinite order;
ii) $x_N x_1^{-1}$ commutes with any $x_j \in F(N)$, where $j > N$.

Proposition 3.4. $C(F(N)) = 1$

Proof. The idea of the proof is similar to the case $N = 2$ which is done in [Gab]. We first show that the group Γ generated by $\gamma := x_N x_1^{-1}$ and x_i, $i > N$ has fixed price $= 1$. Let $\Pi : \Gamma \to \mathrm{Aut}(X, \nu)$ be a free action that generates a (SP1) equivalence relation R_Π of Γ. We prove $C(R_\Pi) = 1$: suffices to show for every $\delta > 0$, $C(R_\Pi) \leq 1 + \delta$. Because γ is of infinite order we can find a sequence A_n of Borel subsets of X such that $\nu(A_n) < \delta/2^n$ and $A_n \cap R_\gamma[x] \neq \emptyset$ a.e. $x \in X$ (see [Gab]). Using ii) above, it is a routine to show that for the following graphing Φ we have $R_\Phi = R_\Pi$:
$\Phi := \{\Pi(\gamma) : X \to X,\ \Pi(x_i)_{|A_i}\ i > N\}$. Next, take Γ_1 the subgroup generated by Γ and x_1. It is easy to see that the set $x_1 \Gamma x_1^{-1} \cap \Gamma$ is infinite (it contains all x_j with $j > N$. Inductively, in N steps we obtain an increasing sequence of subgroups whose union equals $F(N)$. We apply now Critere 3 in [Gab] to conclude that the cost of $F(N)$ is 1. $\square$

The reason all of the above does not work for the subgroup $[F, F]$ is that we do not know the generators of $[F, F]$. We can still start with the element γ and then gradually add elements of $[F, F]$, the idea being to enter the hypotheses of *Critere* 3: however we did not find a way of adding such that to exhaust $[F, F]$.

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